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PLANE RESONANCE ROTATIONS OF A SATELLITE IN AN ELLIPTIC ORBIT

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Stability of plane resonance rotations of a satellite in an elliptic orbit are investigated using the method of Poincaré. It is shown that resonances of the type $k:2$ (k is an integer) are determined by the first approximation with respect to the small parameter, while resonances of type $k:4$ and of types $k:3$ and $k:6$ are determined, respectively, by the second and third approximations. The effect of tidal moments on the existence and stability of plane resonance rotations is investigated.

1. Let us consider the equation of plane oscillations of a satellite in an elliptic orbit [1]

$$(1+e\cos\nu)\frac{d^2\theta}{d\nu^2}-2e\sin\nu\frac{d\theta}{d\nu}+\frac{n^2}{2}\sin 2\theta=2e\sin\nu \qquad (1.1)$$

where θ is the angle between the radius vector of the satellite center of mass and its principal axis of inertia which lies in the orbit plane and corresponds to the moment of inertia C, ν is the true anomaly, e is the orbit eccentricity, $n^2=3(A-C)/B$, and A, B, and C are the principal central moments of inertia of the satellite. We assume that $n^2>0$.

We attribute to motions of class $k:m$ satellite rotations of the form [2]

$$\theta=\nu(k-m)/m+\varphi(\nu) \qquad (1.2)$$

where k is an integer, m is a natural number and k and m are relatively prime numbers; $\varphi(\nu)$ is a periodic function of period $2\pi m$. In the course of ms turns around the planet the satellite in motion (1.2) makes ks turns about an axis perpendicular to the plane of orbit.

Equation (1.1) was the subject of detailed investigation (see, e.g., [2–10] *). Resonaces of the type $k:2$ were revealed in [3–5] by asymptotic methods. Resonances $5:4$ and $7:4$ were numerically determined in [2], and periodic solutions of Eq. (1.1) were investigated in [6–8]. We shall use Poincaré's method of the small parameter [11, 12] for determining resonance motions of the satellite.

We substitute variables

$$x_1=2\theta-2\nu(k-m)/m,\quad x_2=x_1^{\cdot} \qquad (1.3)$$

*) Sarychev, V. A. and Zlatoustov, V. A., Periodic oscillations of a satellite in the elliptical orbit plane. Preprint Inst. Prikladnoi Matematiki Akad. Nauk, SSSR, No. 48, Moscow, 1975.

and introduce the small parameter $\mu = -n^2$, assuming that $(A - C) / B \ll 1$.

The substitution (1.3) transforms Eq. (1.1) into a quasi-linear system with periodic coefficients of period 2π of the form

$$\frac{dx_1}{d\nu} = x_2, \quad (1 + e\cos\nu)\frac{dx_2}{d\nu} = 2e\sin\nu\left(x_2 + 2 + \frac{p}{q}\right) + \mu\sin\left(\frac{p}{q}\nu + x_1\right) \tag{1.4}$$

where p is an integer, q is a natural number, p and q are relatively prime numbers, and $p / q = 2(k - m) / m$.

The satellite resonance motions are determined by the periodic solution of system (1.4) of period $2\pi q$, which for $\mu = 0$ will be called the generating system. It has a periodic (generating) solution x_1°, x_2° of 2π-period which depends on the arbitrary parameter $\alpha \in [0, 2\pi)$ of the form

$$x_1^\circ(\nu, \alpha) = \alpha + (p / q + 2)(\tau(\nu) - \nu) \tag{1.5}$$

$$x_2^\circ(\nu) = \left(\frac{p}{q} + 2\right)\left(\frac{(1 - e^2)^{3/2}}{(1 + e\cos\nu)^2} - 1\right)$$

where $\tau(\nu) = \omega_0 t$ is the dimensionless time and ω_0 is the mean motion of the satellite.

Let us fix the arbitrary value of parameter α. The periodic (of $2\pi q$-period) solution of system (1.4) with initial conditions

$$x_1 = x_1^\circ(0, \alpha) + \beta_1 = \alpha + \beta_1, \quad x_2 = x_2^\circ(0) + \beta_2$$

will be sought in the form

$$x_l = x_l(\mu, \beta_1, \beta_2, \alpha, \nu), \quad l = 1, 2 \tag{1.6}$$

where $\beta_1(\mu, \alpha)$ and $\beta_2(\mu, \alpha)$ are functions of parameter μ that vanish when $\mu = 0$. In accordance with Poincaré's theorem we represent solution (1.6) in the form of series

$$x_l = x_l^\circ + \sum_{k=0}^{\infty}\mu^k\sum_{i,j=0}^{\infty}a_{kij}^{(l)}\beta_1^i\beta_2^j, \quad k^2 + i^2 + j^2 \neq 0; \ l = 1, 2 \tag{1.7}$$

where the coefficients $a_{kij}^{(l)}$ are determined from the solution of the problem

$$\frac{da_{kij}^{(1)}}{d\nu} = a_{kij}^{(2)}, \quad \frac{da_{kij}^{(2)}}{d\nu} = \frac{2e\sin\nu}{1 + e\cos\nu}a_{kij}^{(2)} + f_{kij} \tag{1.8}$$

$$a_{010}^{(1)}(0) = 1, \quad a_{001}^{(2)}(0) = 1$$

$$a_{kij}^{(1)}(0) = 0, (k, i, j) \neq (0, 1, 0); \quad a_{kij}^{(2)}(0) = 0, (k, i, j) \neq (0, 0, 1)$$

where f_{kij} are polynomials are determined by preceding approximations with respect to μ. System (1.8) is integrated in quadratures. In the zero approximation with respect to μ we have

$$a_{010}^{(1)} = 1, \ a_{001}^{(1)} = (1 + e)^2(1 - e^2)^{-3/2}\tau(\nu) \tag{1.9}$$

$$a_{001}^{(2)} = [(1 + e) / (1 + e\cos\nu)]^2$$

$$a_{0jm}^{(1)} = 0, \quad (j, m) \neq (0, 1), (1, 0)$$

$$a_{0jm}^{(2)} = 0, \quad (j, m) \neq (0, 1)$$

The condition of periodicity of solution (1.6) is of the form

$$\psi_l(\mu, \beta_1, \beta_2, \alpha) = [x_l] = 0, \quad l = 1,2$$
$$[x] = x|_{v=2\pi q} - x|_{v=0}$$

Taking into consideration (1.9) we obtain

$$\psi_l = \sigma_l + \mu \sum_{k,i,j=0}^{\infty} [a_{k+1,i,j}^{(l)}] \mu^k \beta_1^i \beta_2^j = 0 \tag{1.10}$$

$$\sigma_1 = [a_{001}^{(1)}]\beta_2, \ \sigma_2 = 0$$

We determine β_1 and β_2 using system (1.10) under condition that $\beta_1(0, \alpha) = 0$ and $\beta_2(0, \alpha) = 0$. It follows from (1.10) that the Jacobian $\{\partial(\psi_1, \psi_2) / \partial(\beta_1, \beta_2)\}_{\mu=0} = 0$ and

$$(\partial\psi_1 / \partial\beta_2)_{\mu=0} = [a_{001}^{(1)}] > 0 \tag{1.11}$$

Taking into account (1.11) we solve the first equation of system (1.10) for β_2. We have

$$\beta_2 = \beta_2(\mu, \beta_1, \alpha), \quad \beta_2(0, 0, \alpha) = 0 \tag{1.12}$$

From (1.12) and the second equation of system (1.10) for the determination of β_1 we obtain an equation of the form [12]

$$\psi_2(\mu, \beta_1, \beta_2(\mu, \beta_1, \alpha), \alpha) = \mu^s (P_s(\alpha) + (dP_s / d\alpha)\beta_1 + O(\mu, \beta_1^2)) = 0 \tag{1.13}$$

where s is some positive integer.

The solution of Eq. (1.13) for β_1 exists when condition

$$P_s(\alpha) = 0 \tag{1.14}$$

is satisfied.

To each parameter $\alpha = \alpha^*$ determined by (1.14) corresponds solution $\beta_1(\alpha^*, \mu)$, that is unique and analytic with respect to μ, when α^* is a simple solution of Eq. (1.14), i.e.

$$(dP_s / d\alpha)_{\alpha=\alpha^*} \neq 0 \tag{1.15}$$

Substituting the obtained values of β_1 and β_2 into (1.7) we obtain for each parameter $\alpha = \alpha^*$ a unique periodic (of $2\pi q$-period) solution of system (1.4), which for $\mu = 0$ becomes the generating solution.

2. Let us investigate the stability of the obtained solution $x_1(v, \alpha^*)$, $x_2(v, \alpha^*)$, and consider for (1.4) a system of equations in variations whose characteristic equation in the Poincaré form is [11]

$$\det \| X_{ij} \|_{\beta_l = \beta_l(\mu, \alpha^*)} = 0, \ l = 1, 2; \ i, j = 1, 2 \tag{2.1}$$
$$X_{ii} = 1 - \rho + \partial\psi_i / \partial\beta_i; \ X_{ij} = \partial\psi_i / \partial\beta_j \quad (i \neq j)$$

which for $\mu = 0$ has the root $\rho = 1$ of multiplicity two, to which corresponds a second order elementary divisor. According to [13] the roots of Eq. (2.1) are to be sought in this case in the form of series in powers of $|\mu|^{1/2}$. We shall seek a solution of the characteristic equation of the form

$$\rho = 1 + |\mu|^{r/2} \rho_1 + |\mu|^{r/2} K(\mu) \tag{2.2}$$

where $K(0) = 0$ and r is some positive integer. Assuming that $\rho_1 \neq 0$ and taking into account (2.2), we write Eq. (2.1) in the form

$$|\mu|^r (\rho_1 + K(\mu))^2 - |\mu|^{r/2} I_* (\rho_1 + K(\mu)) + J_* = 0 \tag{2.3}$$

$$I_* = (\partial\psi_1 / \partial\beta_1 + \partial\psi_2 / \partial\beta_2)_{\beta_i=\beta_i(\mu,\, \alpha^*)}$$

$$J_* = \left(\frac{\partial\psi_1}{\partial\beta_1}\frac{\partial\psi_2}{\partial\beta_2} - \frac{\partial\psi_1}{\partial\beta_2}\frac{\partial\psi_2}{\partial\beta_1}\right)_{\beta_i=\beta_i(\mu,\, \alpha^*)}$$

For stability of the periodic solution it is necessary in this case that $|\rho| = 1$, i.e. ρ_1 is pure imaginary. We shall show that in the case of stable motions the quantity I_* is of an order not higher than r with respect to μ. Let $I_* = |\mu|^a \Psi(\mu)$, where $\Psi(0) \neq 0$, $a \leqslant r$. It then follows from (2.3) that either $\rho_1 = 0$ or there exists root ρ_1 with a nonzero real part, which contradicts the selection of ρ_1 and the condition of stability. Hence $a > r$. Let us show that

$$J_* = (-1)^{s+1} |\mu|^s \left.\frac{dP_s}{d\alpha}\right|_{\alpha=\alpha^*} \left.\frac{\partial\psi_1}{\partial\beta_2}\right|_{\mu=0} + O(|\mu|^{s+1}) \tag{2.4}$$

It follows from (1.13) that

$$\partial\psi_2 (\mu, \beta_1, \beta_2(\mu, \beta_1, \alpha^*), \alpha^*) / \partial\beta_1 = (-1)^s |\mu|^s (dP_s / d\alpha)_{\alpha=\alpha^*} + O(|\mu|^{s+1}, |\mu|^s \beta_1)$$

On the other hand

$$\frac{\partial\psi_2}{\partial\beta_1} = \left\{\frac{\partial\psi_2}{\partial\beta_1} + \frac{\partial\psi_2}{\partial\beta_2}\frac{\partial\beta_2}{\partial\beta_1}\right\}, \quad \frac{\partial\beta_2}{\partial\beta_1} = -\left\{\frac{\partial\psi_1 / \partial\beta_1}{\partial\psi_1 / \partial\beta_2}\right\}$$

Braces in the last two equalities mean that after partial differentiation, (1.12) is substituted for β_2. Formula (2.4) follows from the last three equalities. Note that (2.3) and (2.4) imply that $r = s$. Equating the coefficients at $|\mu|^s$ in Eq. (2.3) and taking into account (1.11), we obtain the necessary stability condition of the form

$$(-1)^s (dP_s / d\alpha)_{\alpha=\alpha^*} < 0 \tag{2.5}$$

which is analogous to the sufficient condition of stability obtained in [14] for the case when $s = 1$, and Eq. (2.1) with $\mu = 0$ has $|\rho^{(1)}| < 1$ as one root and $-\rho^{(2)} = 1$ as the other.

3. Let us pass now to finding the conditions of existence and stability of periodic solutions of system (1.4) in explicit form and also to the derivation of these solutions. Taking into consideration the results in [12] we seek a periodic solution of system (1.4) in the form of formal series

$$x_i = x_i^\circ(\nu, \alpha) + \sum_{\sigma=1}^{\infty} x_{i\sigma}(\nu, \alpha)\mu^\sigma; \quad i = 1, 2 \tag{3.1}$$

where $x_i^\circ(\nu, \alpha)$ is the generating solution (1.5). If the equations

$$[x_{2m}] = 0, \quad m = 1, 2, \ldots, s-1 \tag{3.2}$$

are identically satisfied with respect to α and equation

$$[x_{2s}] = P_s(\alpha) = 0$$

determines the simple solution $\alpha = \alpha^*$, then system (1,4) has a unique periodic solution which in the neighborhood of $\mu = 0$. is analytic with respect to μ.

From (3.1) and (1.4) we obtain for the determination $x_{l\sigma}$ equations of the type (1.8). Integration of these equations yields the following recurrent relationships:

$$x_{1\sigma} = \frac{1}{(1-e^2)^{3/2}}\left(\tau c_\sigma^{(2)} + \tau \int_0^\nu Y_\sigma \, d\nu - \int_0^\nu \tau Y_\sigma \, d\nu\right) + c_\sigma^{(1)} \tag{3.3}$$

$$x_{2\sigma} = \frac{1}{(1+e\cos\nu)^2}\left(\int_0^\nu Y_\sigma \, d\nu + c_\sigma^{(2)}\right)$$

$$Y_\sigma = F^{(\sigma)}(1+e\cos\nu)^2, \quad F^{(1)} = \{F\}$$

$$F^{(\sigma)} = \sum_{k=1}^{\sigma-1} \frac{1}{k!}\left\{\frac{\partial^k F}{\partial x_1^k}\right\} \sum_{i_1+i_2+\ldots+i_k=\sigma-1} x_{1i_1} x_{1i_2} \ldots x_{1i_k}$$

$$F = \sin(\nu p / q + x_1) / (1 + e\cos\nu)$$

where $c_\sigma^{(l)}$ are constants of integration. The functions in braces are determined by the generating solution (1.5). For the determination of α^* and constants of integration $c_\sigma^{(l)}$ we use the conditions of periodicity of solution $[x_{l\sigma}] = 0$, $l = 1, 2$; $\sigma = 1, 2, \ldots$.

Remark 1. System (1.4) is invariant with respect to the substitution

$$\nu \to \nu - 2\pi q, \quad x_1 \to x_1 + 2\pi k / q, \quad x_2 \to x_2 \tag{3.4}$$

where k is an integer. This means that if $x_1(\nu)$, $x_2(\nu)$ is a solution of system (1.4), then $x_1(\nu - 2\pi k) + 2\pi k / q$, $x_2(\nu - 2\pi k)$ is also its solution. When k is a multiple of q these two solutions are identical, otherwise there are, generally, none.

Remark 2. We introduce the notation $P_\sigma^{pq} = [x_{2\sigma}]$. It follows from (3.4) that

$$P_\sigma^{pq}(\alpha) = \sum_{k=1}^{\sigma+1} A_{k\sigma}^{pq} \sin^{\sigma-k+1}\alpha \cos^{k-1}\alpha, \quad \sigma = 1, 2, \ldots$$

where $A_{k\sigma}^{pq}$ are some constants dependent on the orbit eccentricity.

It was shown by numerical methods for $q = 1, 2, 3$ and $m < q$ that equality (3.2) is identical with respect to α, and for $m = q$ it assumes the form

$$P_q^{pq}(\alpha) = A_q^p \sin q\alpha = 0 \tag{3.5}$$

The dependence of $A_2^p / (4\pi)$ and $A_3^p / (12\pi^2)$ on the eccentricity is shown in Fig. 1 by solid and dash lines, respectively. The solid lines *1*, *2*, *3* correspond to $p = 1, 5, 9$; while the dash lines *1*–*5* relate to $p = 1, 2, 4, -2, -1$. Function $A_1^p(e)$ was obtained in [4]. For small e we have

$$A_1^p \sim e^{|p|}, A_2^p \sim e^{(2|p|-1)}, A_3^{-2} \sim A_3^2 \sim e^2, A_3^{-1} \sim A_3^1 \sim e, \quad A_3^4 \sim e^3 \tag{3.6}$$

Note that $q = 1$ corresponds to resonances of the type $k:2$, $q = 2$ to type $k:4$, and $q = 3$ to types $k:3$ and $k:6$ (k is an integer relatively prime to q). For zero eccentricity $A_q^p \neq 0$ only when $p = 0$ and $q = 1$ (resonance $1:1$ of the Moon type); for remaining p and q the conditions of existence of periodic solutions is identically satisfied, since $A_q^p = 0$. When $e = 0$ system (1.4) is analytically integrable [9]; analysis of its solution leads to the conclusion that motions of the type (1.2) on a circular orbit exist for any p and q; all of them, except the resonance $1 : 1$, are unstable.

We shall consider only such orbits for which $A_q^p \neq 0$. Equation (3.5) has the solution

$$\alpha^* = 0, \pi / q, \ldots, (2q - 1) \pi / q; \qquad q = 1, 2, 3$$

which satisfies condition (1.15).

To every α^* corresponds a unique $2\pi q$ -periodic solution of system (1.4) which for $\mu = 0$ becomes the generating solution. According to Remark 1 it is sufficient to determine solutions that correspond to $\alpha^* = 0$ and $\alpha^* = \pi / q$, since the remaining solutions are derived from these using the substitution (3.4). Thus solutions are divided in two classes that correspond to the indicated values of α^*.

The necessary condition of stability assumes the form

$$(-1)^q q \cos q\alpha^* \quad A_q^p < 0, \; q = 1, 2, 3 \tag{3.7}$$

It follows from (3.7) that when $(-1)^q A_q^p < 0$ the solutions of class $\alpha^* = \pi / q$ are unstable, and that condition (3.7) is satisfied for class $\alpha^* = 0$ solutions. When $(-1)^q A_q^p > 0$ solutions of class $\alpha^* = 0$ are unstable and for class $\alpha^* = \pi / q$ (3.7) is satisfied.

Numerical calculations show that at least for the considered p and condition $e < 0.6$, $(-1)^q A_q^p < 0$, and $p > 0$, $q = 1, 2, 3$, when only solutions of class $\alpha^* = 0$ can be stable.

For negative p and increasing $|p|$ the sign of $(-1)^q A_q^p$ alternates beginning from the positive. For example, for $q = 3$ the quantity $-A_q^p$ is positive for $p = -1, -4, -7$ and negative for $p = -2, -5, -8$. In the first case only solutions of class $\alpha^* = \pi / q$ can be stable, in the second only those of class $\alpha^* = 0$.

Initial values of the periodic solution $x_1(0, \alpha^*)$, $x_2(0, \alpha^*)$ are calculated with any required accuracy for fairly small μ, using formulas

$$x_1(0) = \alpha^* + \sum_{l=1}^{N} c_l^{(1)} \mu^l + O(\mu^{N+1})$$

$$x_2(0) = \left(\frac{p}{q} + 2\right)\left[\frac{(1-e^2)^{3/2}}{(1+e)^2} - 1\right] + \sum_{l=1}^{N} c_l^{(2)} \mu^l + O(\mu^{N+1})$$

where $c_l^{(i)}$ are uniquely determined by the periodicity conditions [11]. Thus

$$c_l^{(2)} = \frac{1}{2\pi q} \int_0^{2\pi q} \tau F^{(l)} (1 + e \cos \nu)\, d\nu, \quad l = 1, 2, 3, \ldots$$

Phase patterns of periodic solutions of system (1.4) are shown in Figs. 2–5 for $\alpha^* = 0$. Fig. 2 corresponds to $\mu = -0.1$ and $e = 0.2$; resonances $5 : 4$, $7 : 4$, and $9 : 4$ correspond to $q = 2$, $p = 1, 3, 5$ and initial conditions $x_2(0) = -0.852, -1.138, -1.490$. The phase pattern for resonance $11 : 4$, appears in Fig. 3 with $q = 2$, $p = 7$, $e = 0.02$, and $\mu = -0.1$; for $x_2(0)$

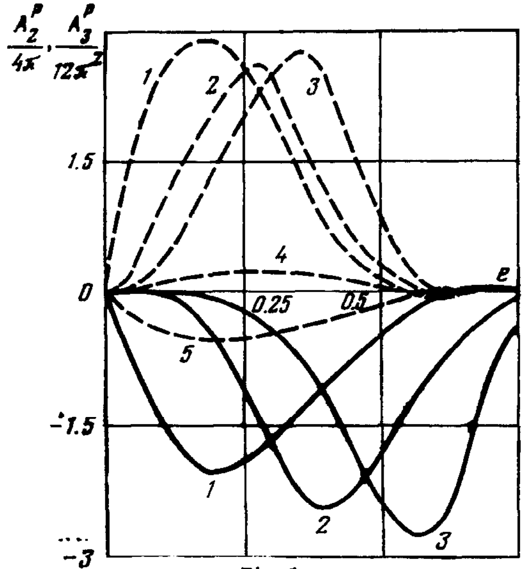

Fig. 1

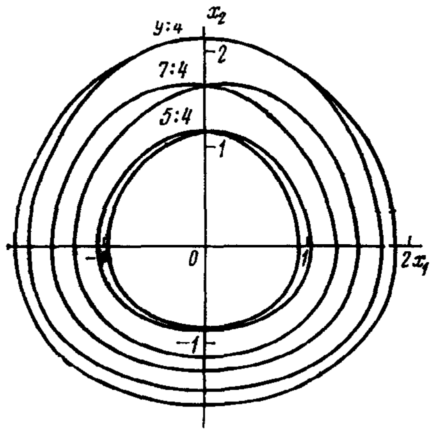

Fig. 2

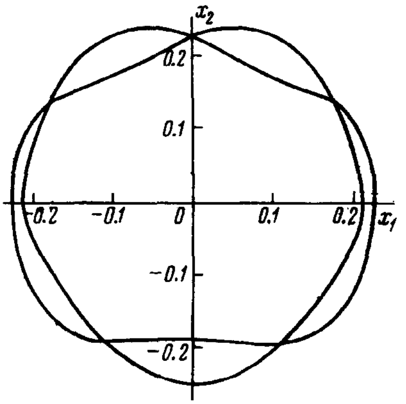

Fig. 3

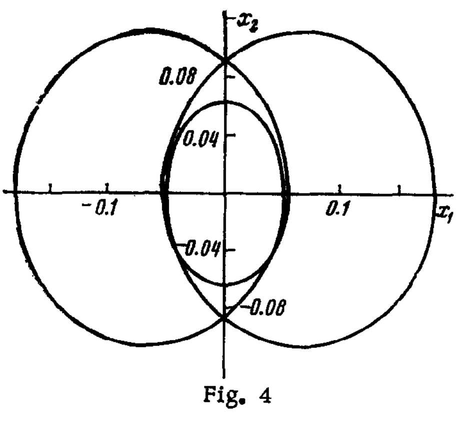

Fig. 4

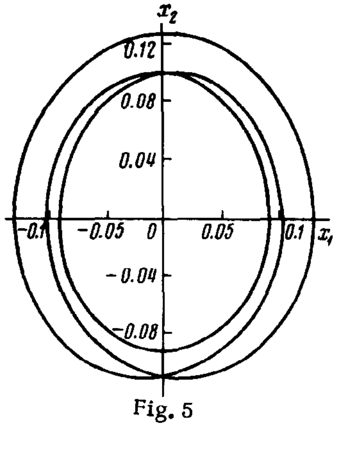

Fig. 5

we have in this case $x_2(0) = -0.186$. Fig. 4 corresponds to resonance $7:6$, $q = 3$, $p = 1$; $e = 0.02$, $\mu = -0.01$, and $x_2(0) = -0.0637$. Resonance of type $4:3$ is shown in Fig. 5 for $q = 3$, $p = 2$, $\mu = -0.01$, $e = 0.02$, and $x_2(0) = -0.0926$.

Initial values of $x_2(0)$ for $\alpha^* = 0$ for certain resonances are also tabulated below. All derived periodic solutions with $\alpha^* = 0$ have $x_1(0) = 0$ as initial value.

q	e	μ	p	$k:m$	$x_2(0)$
2	0.02	—0.1	3	7:4	—0.061
			5	9:4	—0.134
		—0.01	3	7:4	—0.13
			5	9:4	—0.173
3	0.2	—0.01	—1	5:6	—0.601
			1	7:6	—0.8
			2	4:3	—0.93
			4	5:3	—1.029

Phase patterns for $\alpha^* = 2\pi k/q$ are obtained in conformity with Remark 1 from the phase pattern for $\alpha^* = 0$ by the $2\pi k/q$ shift along the x_1-axis.

Let us explain the physical meaning of conditions of stability and existence of periodic solutions (1.14), (1.15), and (2.5) for $q = 1$. Then $P_1(\alpha)$ and the perturbing force function are of the form

$$P_1(\alpha) = \frac{1}{(1+e)^2} \int_0^{2\pi} \sin[p\tau + 2(\tau - \nu) + \alpha](1 + e\cos\nu)\, d\nu$$

$$U = -cR^{-3}\cos(p\nu + x_1),\ c = \text{const},\ c > 0$$

We average U with respect to time using instead of x_1 the generating solution; the mean value of u as a function of parameter α is of the form

$$u(\alpha) = -c' \int_0^{2\pi} \cos[p\tau + 2(\tau - \nu) + \alpha](1 + e\cos\nu)\, d\nu$$

$$c' = \text{const} > 0$$

Comparing P_1 with u, taking into account the stability condition (2.5), we conclude that the periodic (of period 2π) solutions of system (1.4) which become generating solutions when $\mu = 0$ can only correspond to those values of parameter $\alpha = \alpha^*$ for which the time averaged perturbing force function in the generating solution attains an extremal value. Solutions can be stable only when the extremum is a maximum.

The obtained extremum principle is in partial agreement with the hypothesis formulated in [2] according to which the force function averaged along the exact solution of system (1.4) has in conformity with initial data a maximum for stable resonance motions of the form (1.2).

4. Let now the satellite be subjected besides gravitational effects to the moment of tidal forces [15]

$$M = -k\omega / r^6$$

where k is a positive constant, r is the distance from the satellite to the planet, and ω is the angular velocity of satellite rotation relative to the orbital coordinate system. Let us assume that the moment of tidal forces is of order n relative to μ. The presence of such moment results in the appearance in the right-hand side of the second equation of system (1.4) of the additional term

$$-|\mu|^n k_1 (1 + e\cos\nu)^5 (x_2' + p/q) \tag{4.1}$$

where k_1 is a positive constant, and x_1' and x_2' are variables of the problem with a tidal moment.

We seek in this case a periodic solution using the procedure applied to system (1.4). Since system (1.4) with allowance for (4.1) coincides with an accuracy within μ^{n-1} with system (1.4), all results obtained with allowance for (4.1) coincide with an accuracy smaller than μ^n with respective formulas for (1.4). For ψ_2' we have

$$\psi_2'(\mu, \beta_1', \beta_2'(\mu, \beta_1', \alpha'), \alpha') = \psi_2(\mu, \beta_1', \beta_2'(\mu, \beta_1', \alpha'), \alpha') + \mu^n(b + O(\mu, \beta_1')) \tag{4.2}$$

$$b = (-1)^{n+1} \frac{2\pi q k_1}{(1+e)^2} \left\{\left(\frac{p}{q} + 2\right)(1-e^2)^{3/2}\left(1 + 3e^2 + \frac{3}{8}e^4\right) - 2 - 15e^2 - \frac{45}{4}e^4 - \frac{5}{8}e^6\right\}$$

where the prime indicates the presence of the tidal moment.

Let $n < s$. Then Eq. (4.2) is solvable for β_1' ($\beta_1'(0, \alpha') = 0$) only under condition that $b = 0$, an equality that is generally not satisfied (except for a finite number of eccentricities).

When $n = s$ the condition of existence is of the form

$$P_s(\alpha') + b = 0$$

or, if $q = 1, 2, 3$

$$A_q^p \sin q\alpha' + b = 0$$

which is solvable for α' under condition that

$$|b / A_q^p| < 1 \tag{4.3}$$

It was shown above that $s = q$ for $q = 1, 2, 3$ and $s \geqslant 4$ when $q \geqslant 4$. Hence, when the moment of tidal forces is fairly large (sufficient if $n < s$), the condition of existence of solution of Eq. (4.2) is not satisfied, i.e. a periodic solution of system (1.4) with allowance for (4.1) which for $\mu = 0$ becomes the generating solution, does not exist. For considerable q much weaker tidal forces are required for "blurring" the resonance motion than in the case of $q = 1$.

Note that although μ is assumed small for each specific satellite, it is nontheless finite. Consider the case of $q = 1, 2, 3$. Let the magnitude of the tidal moment be of order higher than q with respect to μ. In that case we assume that $M = -|\mu|^q k_2\omega / r^6$, i.e. k_2 is an infinitely small quantity. Let us consider inequality (4.3). It follows from (3.6) that for small eccentricities the quantity $|A_q^p|$, $q = 1,2,3$, rapidly decreases as $|p|$ increases. The quantity b varies relatively little as $|p|$ increases, hence condition (4.3) of existence of a periodic solution that becomes the generating one when $\mu = 0$ is satisfied for small $|p|$ and not satisfied for fairly large $|p|$. Thus for $q = 1, 2, 3$ even small tidal moments blurr almost all resonances, except a finite number of resonances with p small in absolute value .

For natural celestial bodies parameter $\mu \sim 10^{-3} \div 10^{-5}$. If one assumes that tidal forces are 10^6 times lower than gravitational perturbations, then even for such celestial bodies only resonances with $q = 1$, and possibly $q = 2$, are possible when $|p|$ is fairly small and the orbit is slightly elliptic.

Blurring of high-order resonances with the addition of dissipative forces follows in the general case from the theorem on the necessary conditions of synchronism in a dynamic system [16].

Let us consider the case of $n > s$ and p not very large in absolute value in the sence that the resonance is not blurred by tides. The conditions of existence of periodic solution of system (1.4) with allowance for (4.1) that becomes the generating solution when $\mu = 0$ is, then, identical to the similar conditions (1.14) and (1.15) that relate to the case of absence of tides. Hence α'^* is equal α^* . Omitting the proof, we formulate under these conditions the following statement for resonance motions of the type $k : 2$ $(q = 1)$. If the roots $\rho^{(i)}$, $i = 1, 2$ of the characteristic equation (2.1) (in the absence of tides) differ in absolute value from unity by a quantity of order $o(\mu^n)$ and the moment of tidal forces is of order μ^n , a resonance motion with the addition of the tidal moment becomes asymptotically stable. In particular, if the motion is stable without tides, it becomes asymptotically stable when the tidal moment is added.

It can be shown that $|\rho^{(i)}|$, $i = 1, 2$ differ from unity at least by a quantity of order $o(\mu^2)$ and, consequently, if the tidal moment is of order μ^2 , all resonances of the type $k : 2$, for which the conditions of existence are satisfied, are asymptotically stable.

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